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Calculus of Variations

Partial regularity for biharmonic maps, revisited

Michael Struwe

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Abstract Extending our previous results with Tristan Rivière for harmonic maps, we show how partial regularity for stationary biharmonic maps into arbitrary targets can be naturally obtained via gauge theory in any dimensions $m \geq 4$.

1 Introduction

In [9], jointly with Tristan Rivière we presented a new approach to the partial regularity for stationary weakly harmonic maps in dimension $m \geq 2$ as a special case of a regularity result for elliptic systems

$$-\Delta u^i = \Omega^{ij} \cdot \nabla u^j \quad \text{in } B \tag{1}$$

on a ball $B = B^m \subset \mathbb{R}^m$ with $\Omega = (\Omega^{ij}) \in L^2(B, so(n) \otimes \wedge^1\mathbb{R}^m)$ and with $u = (u^1, \ldots, u^n) \in H^1(B, \mathbb{R}^n)$ satisfying the Morrey growth assumption

$$\sup_{x \in B,\, r>0} \left(\frac{1}{r^{m-2}} \int_{B_r(x) \cap B} (|\nabla u|^2 + |\Omega|^2)\, dx \right)^{1/2} < \varepsilon(m). \tag{2}$$

A key ingredient in this new approach is the natural use of gauge theory, which is motivated by the anti-symmetry of the 1-form $\Omega = \Omega^{ij}$. Previously, Rivière [8] already had recognized this structure as the essential structure of the harmonic map system in $m = 2$ space dimensions, allowing him to obtain an equivalent formulation of this equation in divergence form. His results generalize to a large number of conformally invariant equations of second order. Subsequently, Lamm and Rivière [6] obtained a similar equivalent formulation of the biharmonic map system as a conservation law in the *"conformal"* case of $m = 4$ space dimensions. However, just as the methods of [8] no longer seem applicable when $m > 2$, also the approach in [6] seems to fail in dimensions $m > 4$.

M. Struwe (✉)
Mathematik, ETH-Zürich, 8092 Zürich, Switzerland
e-mail: struwe@math.ethz.ch

Our aim in this short note is to extend the approach in [9] to fourth order equations and to recover the known partial regularity results for stationary (extrinsic) biharmonic maps into an arbitrary closed target manifold $N \subset \mathbb{R}^n$ by a simpler method and under minimal regularity assumptions. In particular, we show the following result which improves the pioneering work of Chang et al. [3] and the later results by Wang [14] and Strzelecki [10] in this regard.

Theorem 1.1 *Let $N^k \subset \mathbb{R}^n$ be a closed submanifold of class C^3. Let $m \geq 4$ and suppose $u \in H^2(B; N)$ is a stationary biharmonic map on a ball $B = B^m \subset \mathbb{R}^m$. There exists a constant $\varepsilon_0 > 0$ depending only on N and m with the following property. Whenever on some ball $B_R(x_0) \subset B$ there holds*

$$R^{4-m} \int_{B_R(x_0)} (|\nabla^2 u|^2 + |\nabla u|^4) dx < \varepsilon_0, \tag{3}$$

then u is Hölder continuous (and hence as smooth as the target permits) on $B_{R/3}(x_0)$. In particular, u is smooth off a set $S \subset B$ of vanishing $(m-4)$-dimensional Hausdorff measure.

In the following section we first derive a useful form of the biharmonic map equation. In Sect. 3 we give an overview of Morrey spaces and recall the interpolation results and results from gauge theory that we need. Finally, we present the proof of Theorem 1.1 in Sect. 4. It would be interesting to see if our method can be extended to general linear systems of fourth order that exhibit a structure similar to the one of Eq. (4) below, as is the case for second order systems (1) or in the *"conformal"* case $m = 4$ considered in [6].

2 Biharmonic maps

In a first step we cast the equation for a biharmonic map $u \in H^2(B, \mathbb{R}^n)$ into the form

$$\Delta^2 u = \Delta(D \cdot \nabla u) + \operatorname{div}(E \cdot \nabla u) + F \cdot \nabla u \quad \text{in} \ \ B \tag{4}$$

previously considered in [6] in dimension $m = 4$. In contrast to [6], however, here we decompose the function F as $F = G + \Delta\Omega$ with $\Omega = (\Omega^{ij}) \in H^1(B, so(n) \otimes \wedge^1 \mathbb{R}^m)$. The coefficient functions D, E, G, and Ω naturally depend on u and satisfy the growth conditions

$$\begin{aligned} |D| + |\Omega| &\leq C|\nabla u|, \\ |E| + |\nabla D| + |\nabla \Omega| &\leq C|\nabla^2 u| + C|\nabla u|^2, \\ |G| &\leq C|\nabla^2 u||\nabla u| + C|\nabla u|^3. \end{aligned} \tag{5}$$

To see (4) consider for simplicity the case of a biharmonic map $u = (u^1, \dots, u^n)$ to a closed hypersurface $N \subset \mathbb{R}^n$ with normal ν. As in [9], the general case is obtained in similar fashion with the help of a smooth local orthonormal frame $\nu_1, \dots, \nu_k$ for the normal bundle along N. Denoting as $\pi_N : U \subset \mathbb{R}^n \to N$ the projection from a tubular neighborhood U of N onto N and letting $w = \nu \circ u$, then $d\pi_N(u) = id - w \otimes w : \mathbb{R}^n \to T_u N$ is the projector onto the tangent space along the map u.

From the variational characterization of weakly biharmonic maps $u \in H^2(B, N)$ we have

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \left(\int_B |\Delta \pi_N(u + \varepsilon\varphi)|^2 \, dx \right) = 2 \int_B \Delta u \cdot \Delta(d\pi_N(u)\varphi) \, dx \tag{6}$$

for all $\varphi \in C_0^\infty(B, \mathbb{R}^n)$. Hence we may write the biharmonic map system as

$$\begin{aligned}0 &= d\pi_N(u)\Delta^2 u = \Delta\left(d\pi_N(u)\Delta u\right) - 2\nabla(d\pi_N(u)) \cdot \nabla\Delta u - \Delta\left(d\pi_N(u)\right)\Delta u\\ &= \Delta^2 u - \Delta((w\otimes w)\Delta u) + 2\nabla(w\otimes w)\cdot\nabla\Delta u + \Delta(w\otimes w)\Delta u. \end{aligned} \tag{7}$$

Observing that $w^j\nabla u^j = 0$, following Hélein [5] we rewrite

$$((w\otimes w)\Delta u)^i = w^i w^j \Delta u^j = -w^i \nabla w^j \cdot \nabla u^j = (w^j\nabla w^i - w^i\nabla w^j)\cdot\nabla u^j, \tag{8}$$

where we tacitly sum over repeated indices.

Similarly, we have

$$\begin{aligned}(\nabla(w\otimes w)\cdot\nabla\Delta u)^i &= \nabla(w^i w^j)\cdot\nabla\Delta u^j\\ &= \Delta(\nabla(w^i w^j)\cdot\nabla u^j) - 2\nabla^2(w^i w^j)\cdot\nabla^2 u^j - \nabla\Delta(w^i w^j)\cdot\nabla u^j\\ &= \Delta(w^i\nabla w^j\cdot\nabla u^j) - 2\mathrm{div}(\nabla^2(w^i w^j)\cdot\nabla u^j) + \nabla\Delta(w^i w^j)\cdot\nabla u^j. \end{aligned} \tag{9}$$

Finally, we have

$$(\Delta(w\otimes w)\Delta u)^i = \Delta(w^i w^j)\Delta u^j = \mathrm{div}(\Delta(w^i w^j)\nabla u^j) - \nabla\Delta(w^i w^j)\cdot\nabla u^j, \tag{10}$$

and from (7) we obtain the equation

$$\begin{aligned}\Delta^2 u^i &= 3\Delta((w^j\nabla w^i - w^i\nabla w^j)\cdot\nabla u^j) + 4\mathrm{div}(\nabla^2(w^i w^j)\cdot\nabla u^j)\\ &\quad - \mathrm{div}(\Delta(w^i w^j)\nabla u^j) - \nabla\Delta(w^i w^j)\cdot\nabla u^j. \end{aligned} \tag{11}$$

This equation has the structure (4), that is, in components,

$$\Delta^2 u^i = \Delta\left(D_\alpha^{ij}\partial_\alpha u^j\right) + \partial_\alpha\left(E_{\alpha\beta}^{ij}\partial_\beta u^j\right) + F_\alpha^{ij}\partial_\alpha u^j \quad \text{in } B.$$

Indeed, we may let

$$D_\alpha^{ij} = 3(w^j\partial_\alpha w^i - w^i\partial_\alpha w^j),\ E_{\alpha\beta}^{ij} = 4\partial_\alpha\partial_\beta(w^i w^j) - \delta_{\alpha\beta}\Delta(w^i w^j), \tag{12}$$

satisfying the estimates

$$|D| \le C|\nabla u|,\ |E| + |\nabla D| \le C|\nabla^2 u| + C|\nabla u|^2. \tag{13}$$

For the remaining term we once more use the identity $w^j\nabla u^j = 0$ to expand

$$\begin{aligned}\nabla\Delta\ (w^i w^j)\cdot\nabla u^j &= (w^i\nabla\Delta w^j - w^j\nabla\Delta w^i)\cdot\nabla u^j - G_1^{ij}\cdot\nabla u^j\\ &= \Delta(w^i\nabla w^j - w^j\nabla w^i)\cdot\nabla u^j - G^{ij}\cdot\nabla u^j, \end{aligned} \tag{14}$$

where the coefficient functions G_1 and G involve sums of terms like $\nabla^2 w^i\nabla w^j$. Hence these functions may be estimated

$$|G| + |G_1| \le C|\nabla u||\nabla^2 u| + C|\nabla u|^3. \tag{15}$$

Finally, we let

$$\Omega^{ij} = (w^i dw^j - w^j dw^i), \quad 1 \le i, j \le n, \tag{16}$$

satisfying

$$|\Omega| \le C|\nabla u|,\ |\nabla\Omega| \le C|\nabla^2 u| + C|\nabla u|^2. \tag{17}$$

Note that the constants C in (13), (15), and (17) only depend on a C^2-bound for ν and hence may be chosen uniformly for a closed manifold N of class C^3.

Finally, recall that a weakly biharmonic map u is called stationary if it also is a critical point for the Hessian energy with respect to variations of the form $u\circ(id + \varepsilon\tau)$, where $\tau \in C_0^\infty(B; \mathbb{R}^m)$, $|\varepsilon| << 1$.

3 Morrey spaces and gauge theory

Recall that for any $1 \leq p < \infty$ and any $s < m$ a function $f \in L^p(B)$ belongs to the homogeneous Morrey space $L^{p,s}(B)$ on a ball $B \subset \mathbb{R}^m$, provided that

$$\|f\|^p_{L^{p,s}(B)} = \sup_{x_0 \in B\,,\, r>0} \left(\frac{1}{r^s} \int_{B_r(x_0)\cap B} |f|^p dx \right) < \infty\,, \tag{18}$$

and $f \in \mathrm{BMO}(B)$, if

$$[f]^p_{\mathrm{BMO}(B)} = \sup_{x_0 \in B\,,\, r>0} \left(r^{-m} \int_{B_r(x_0)\cap B} |f - \bar{f}_{r,x_0}|^p \right) < \infty\,, \tag{19}$$

where

$$\bar{f}_{r,x_0} = \fint_{B_r(x_0)\cap B} f\, dx, \tag{20}$$

denotes the average of f on the set $B_r(x_0) \cap B$. Note that Hölder's inequality for $1 \leq p \leq q < m$ implies the bound

$$\|f\|_{L^{p,m-p}(B)} \leq \|f\|_{L^{q,m-q}(B)} \tag{21}$$

for any $f \in L^{q,m-q}(B)$.

For $k \in \mathbb{N}$ and $s = m - kp$ we also use the notation $f \in L^{p,m-kp}_k(B)$, provided that $f \in W^{k,p}(B)$ with $\nabla^l f \in L^{p,m-lp}(B)$ for $0 < l \leq k$. For any $f \in L^{p,m-kp}_k(B)$ Poincaré's inequality

$$\int_{B_r(x_0)\cap B} |f - \bar{f}_{r,x_0}|^p\, dx \leq Cr^p \int_{B_r(x_0)\cap B} |\nabla f|^p\, dx \tag{22}$$

then implies the bound

$$[f]_{\mathrm{BMO}(B)} \leq C||\nabla f||_{L^{p,m-p}(B)}. \tag{23}$$

An important role in our proof of Theorem 1.1 is played by the following refinement of the Gagliardo–Nirenberg interpolation result, due to Adams and Frazier [1]. A very elegant proof using $\mathcal{H}^1$-BMO duality was later given by Strzelecki [11].

Proposition 3.1 *For any $s > 1$ there exists a constant C such that for any $u \in W^{2,s} \cap \mathrm{BMO}(\mathbb{R}^m)$ with compact support there holds*

$$||\nabla u||^2_{L^{2s}(\mathbb{R}^m)} \leq C[u]_{\mathrm{BMO}(\mathbb{R}^m)}||\nabla^2 u||_{L^s(\mathbb{R}^m)}.$$

With the help of (23) Proposition 3.1 may be localized and scaled to yield the following estimate in Morrey norms. A similar result is stated in [14, Proposition 4.3].

Proposition 3.2 *For any $1 < s \leq m/2$ there exists a constant C such that for any ball $B \subset \mathbb{R}^m$ and any $u \in L^{s,m-2s}_2(B)$ there holds*

$$||\nabla u||^2_{L^{2s,m-2s}(B)} \leq C||\nabla u||_{L^{1,m-1}(B)}(||\nabla^2 u||_{L^{s,m-2s}(B)} + ||\nabla u||_{L^{s,m-s}(B)}).$$

Since the argument is somewhat delicate we briefly present the proof of Proposition 3.2 in Appendix A.

With these prerequisites we can now state the results from gauge theory that we need for dealing with Eq. (16). As shown by Meyer and Rivière [7, Theorem I.3], and Tao and Tian [12, Theorem 4.6], the results from Uhlenbeck [13] on the existence of Coulomb gauges may be extended to connections in suitable Morrey spaces. We state their result on an arbitrary ball $B \subset \mathbb{R}^m$; all norms refer to B. In order to emphasize the Coulomb gauge condition, we write the gauge-equivalent connection 1-form as $*d\xi$.

Lemma 3.3 *There exists $\varepsilon = \varepsilon(m,n) > 0$ and $C > 0$ with the following property: For every $\Omega \in L_1^{2,m-4} \cap L^{4,m-4}(B, so(n) \otimes \wedge^1\mathbb{R}^m)$ with*

$$||\nabla\Omega||_{L^{2,m-4}} + ||\Omega||^2_{L^{4,m-4}} \leq \varepsilon(m,n) \tag{24}$$

there exist $P \in H^2(B; SO(n))$ and $\xi \in H^2(B, so(n) \otimes \wedge^{m-2}\mathbb{R}^m)$ such that

$$dPP^{-1} + P\Omega P^{-1} = *d\xi \quad \text{on } B \tag{25}$$

and

$$d(*\xi) = 0 \quad \text{on } B,\ \xi\,|_{\partial B} = 0. \tag{26}$$

In addition, we have $P,\ \xi \in L_2^{2,m-4}(B)$ with

$$\begin{aligned} &||\nabla^2 P||_{L^{2,m-4}} + ||\nabla P||_{L^{2,m-2}} + ||\nabla^2\xi||_{L^{2,m-4}} + ||\nabla\xi||_{L^{2,m-2}} \\ &\quad \leq C(||\nabla\Omega||_{L^{2,m-4}} + ||\Omega||^2_{L^{4,m-4}}) \leq C\varepsilon(m,n). \end{aligned} \tag{27}$$

Note that via Proposition 3.2 from (27) we also obtain that $P,\ \xi \in L_1^{4,m-4}(B)$ with

$$||\nabla P||_{L^{4,m-4}} + ||\nabla\xi||_{L^{4,m-4}} \leq C(||\nabla\Omega||_{L^{2,m-4}} + ||\Omega||^2_{L^{4,m-4}}) \leq C\varepsilon(m,n). \tag{28}$$

4 Proof of Theorem 1.1

Throughout the following we assume that condition (3) is satisfied on $B_3(0)$ for some number $\varepsilon_0 = \varepsilon_0(m,N) > 0$ to be determined in the sequel. As was shown in [3, Lemma 4.8], or [14, Lemma 5.3], for a stationary biharmonic map this implies the Morrey bound

$$\varepsilon_1^4 := ||\nabla^2 u||^2_{L^{2,m-4}(B_2(0))} + ||\nabla u||^4_{L^{4,m-4}(B_2(0))} < C\varepsilon_0; \tag{29}$$

with a constant $C = C(N,m)$. Clearly we may assume that $\varepsilon_1 \leq 1$. The bound (29) is a consequence of the monotonicity inequality for stationary biharmonic maps. The latter was formally derived by Chang et al. [3], Proposition 3.2; for stationary biharmonic maps of class H^2 a rigorous derivation of this key result was later given by Angelsberg [2]. In Appendix B we show how the bound (29) may be derived from the monotonicity inequality directly, without further use of the biharmonic map system. This result may be of independent interest.

As in [9] we interpret the 1-form $\Omega \in H^1(B; so(n) \otimes \wedge^1\mathbb{R}^n)$ arising in Eq. (4) as a connection in the $SO(n)$-bundle $u^*T\mathbb{R}^n$. Taking account of (17) and (29), from Lemma 3.3 for sufficiently small $\varepsilon_0 = \varepsilon_0(N,m) > 0$ we can find a gauge transformation P, transforming Ω into Coulomb gauge. Applying the gauge transformation P to Δu, in a first step we obtain

$$\begin{aligned} P\Delta^2 u + \nabla\Delta P \cdot \nabla u &= \mathrm{div}\,(\nabla(P\Delta u) - 2\nabla P\Delta u + \Delta P\nabla u) \\ &= \Delta(P\Delta u) - 2\mathrm{div}^2(\nabla P \otimes \nabla u) + \mathrm{div}(2\nabla^2 P \cdot \nabla u + \Delta P\nabla u), \end{aligned} \tag{30}$$

where we let $\operatorname{div}^2(\nabla P \otimes \nabla u) = \partial_\alpha \partial_\beta(\partial_\alpha P\, \partial_\beta u)$ for short. Observing the identity

$$\begin{aligned} P\,(\Delta(D \cdot \nabla u) + \operatorname{div}(E \cdot \nabla u) + F \cdot \nabla u) &= \Delta(PD \cdot \nabla u) \\ &+ \operatorname{div}\left((PE - 2\nabla P D) \cdot \nabla u\right) + (\Delta P D + PF - \nabla P \cdot E) \cdot \nabla u, \end{aligned} \tag{31}$$

from (4) and (30) then we find

$$\begin{aligned} \Delta(P \Delta u) &= \Delta(PD \cdot \nabla u) + 2\operatorname{div}^2(\nabla P \otimes \nabla u) - \operatorname{div}\left(2\nabla^2 P \cdot \nabla u + \Delta P \nabla u\right) \\ &+ \operatorname{div}\left((PE - 2\nabla PD) \cdot \nabla u\right) + (\nabla \Delta P + \Delta P D + PF - \nabla P \cdot E) \cdot \nabla u. \end{aligned} \tag{32}$$

Letting

$$\begin{aligned} (D_P)^{ik}_\alpha &= \delta_{\alpha\beta} P^{ij} D^{jk}_\beta + 2\partial_\alpha P^{ik}, \\ (E_P)^{ik}_{\alpha\beta} &= P^{ij} E^{jk}_{\alpha\beta} - 2\partial_\alpha P^{ij} D^{jk}_\beta - \delta_{\alpha\beta} \Delta P^{ik} - 2\partial_\alpha \partial_\beta P^{ik} \end{aligned} \tag{33}$$

and setting

$$\begin{aligned} G_P &= \nabla \Delta P + \Delta P D + PF - \nabla P \cdot E - {*d}\Delta\xi P \\ &= \nabla \Delta P + \Delta P D + PG + P\Delta\Omega - \nabla P \cdot E - \Delta((\nabla P + P\Omega)P^{-1})P, \end{aligned} \tag{34}$$

we finally obtain the gauge-equivalent form

$$\Delta(P\Delta u) = \operatorname{div}^2(D_P \otimes \nabla u) + \operatorname{div}(E_P \cdot \nabla u) + G_P \cdot \nabla u + {*d}\Delta\xi \cdot P\nabla u \tag{35}$$

of Eq. (4), where in view of the cancellations in (34) we have

$$\begin{aligned} |D_P| &\le C(|\nabla u| + |\nabla P|), \\ |\nabla D_P| + |E_P| &\le C(|\nabla^2 u| + |\nabla u|^2 + |\nabla^2 P| + |\nabla P|^2), \\ |G_P| &\le C(|\nabla^2 u| + |\nabla^2 P|)(|\nabla u| + |\nabla P|) + C(|\nabla u|^3 + |\nabla P|^3). \end{aligned} \tag{36}$$

We regard (35) and (25) as a coupled system of equations for u and P.

Fix numbers $1 < p < m/2 < q < m$ with $1/p + 1/q = 1$. Our aim in the following is to derive a Morrey-type decay estimate

$$\int_{B_r(x_0)} |\nabla u|^p \, dx \le C r^{m-p+\alpha p} \tag{37}$$

for all $x_0 \in B_1(0)$ and all $0 < r < 1$ with uniform constants C and $\alpha > 0$. By Morrey's Dirichlet growth theorem then $u \in C^{0,\alpha}(B_1(0))$, as claimed.

Fix $x_0 \in B_1(0)$. For $0 < r < 1$ define

$$\Psi_1(u;r) = ||\nabla u||^p_{L^{p,m-p}(B_r(x_0))},\ \Psi_2(u;r) = ||\Delta u||^p_{L^{p,m-2p}(B_r(x_0))},$$

and similarly for P. For a suitable constant $0 < \gamma < 1$ to be determined below then we let

$$\Psi(u;r) = \Psi_1(u;r) + \gamma^{-m}\Psi_2(u;r),$$

and likewise for P. Finally, with the constant $C_2 \ge 1$ determined in Lemma 4.4 below we let

$$\Psi(r) = \Psi(P;r) + C_2\Psi(u;r).$$

For the proof of (37) then it suffices to show that for all $r < 1$ we can bound

$$\Psi(r) \le C r^{p/4} \tag{38}$$

with a constant C independent of x_0 and r. In view of (29), moreover, we only need to verify (38) for $r < \gamma^2$.

Our argument will also involve scaled estimates for $\nabla^2 u$. Note that we can estimate $\nabla^2 u$ in terms of Δu by means of the Calderòn–Zygmund inequality.

Proposition 4.1 *For any $s > 1$, any $0 < \gamma < 1$ there exists a constant C such that for any $R > 0$ and any $u \in W^{2,s} \cap W_0^{1,s}(B_R(0))$ there holds*

$$||\nabla^2 u||_{L^{s,m-2s}(B_{\gamma R}(0))} \le C \left(||\Delta u||_{L^{s,m-2s}(B_R(0))} + ||\nabla u||_{L^{s,m-s}(B_R(0))} \right).$$

On any ball $B_R(x_1) \subset B_{R_0}(x_0)$ with $0 < R_0 < \gamma$ we split

$$P\Delta u = f + h, \tag{39}$$

where $\Delta h = 0$ in $B_R(x_1)$ and where $f \big|_{\partial B_R(x_1)} = 0$ in the weak sense (so that (44) below holds true).

Our first task now is to show that the component f in this decomposition is of *"lower order"* in the following sense.

Lemma 4.2 *With a uniform constant C there holds*

$$R^{2p-m} \int_{B_R(x_1)} |f|^p \, dx \le C\varepsilon_1 \Psi(\gamma^{-1} R_0). \tag{40}$$

Proof By scale invariance of the expressions, we may scale so that $B_R(x_1) = B_1(0)$. In a first step we establish the estimate

$$||f||_{L^p} \le C\varepsilon_1 (||\nabla u||_{L^{2p}} + ||\nabla P||_{L^{2p}} + ||\nabla u||^2_{L^4} + ||\nabla P||^2_{L^4}). \tag{41}$$

Here and in the following computations all norms refer to the domain $B = B_1(0)$.

To see (41), note that by duality we have

$$||f||_{L^p} \le C \sup_{\varphi \in L^q(B)); ||\varphi||_{L^q} \le 1} \int_B f\varphi \, dx. \tag{42}$$

For any $\varphi \in L^q(B)$ with $||\varphi||_{L^q} \le 1$ denote as Φ the solution to the Dirichlet problem $\Delta\Phi = \varphi$ on B with $\Phi = 0$ on ∂B. By the Calderòn–Zygmund inequality and Sobolev's embedding then with $q^* > m$ satisfying $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{m}$ we have $\Phi \in W^{2,q} \cap W_0^{1,q^*}(B)) \hookrightarrow C^{1-m/q^*}(B)$, and there holds

$$||\Phi||_{L^\infty} \le C||\Phi||_{W^{2,q}} \le C||\varphi||_{L^q} \le C. \tag{43}$$

Hence we deduce that

$$||f||_{L^p} \le C \sup_{\Phi \in W^{2,q} \cap W_0^{1,q^*}(B)); ||\Phi||_{W^{2,q}} \le 1} \int_B f \Delta\Phi \, dx.$$

For any such Φ we now integrate by parts and use (35) to split

$$\int_B f\Delta\Phi \, dx = \int_B \Delta(P\Delta u)\Phi \, dx = I + II + III + IV. \tag{44}$$

By (27)–(29) and (36) the terms

$$\begin{aligned} I + II &= \int_B \operatorname{div}\left(\operatorname{div}(D_P \otimes \nabla u) + (E_P \cdot \nabla u)\right) \Phi \, dx \\ &= -\int_B (\operatorname{div}(D_P \otimes \nabla u) + (E_P \cdot \nabla u)) \cdot \nabla\Phi \, dx \end{aligned}$$

can be estimated

$$
\begin{aligned}
I + II &\leq C \int_B (|\nabla^2 u| + |\nabla^2 P| + |\nabla u|^2 + |\nabla P|^2)(|\nabla u| + |\nabla P|)|\nabla \Phi|\,dx \\
&\leq C(||\nabla^2 u||_{L^2} + ||\nabla^2 P||_{L^2} + ||\nabla u||_{L^4}^2 + ||\nabla P||_{L^4}^2) \\
&\quad \times (||\nabla u||_{L^{2p}} + ||\nabla P||_{L^{2p}})||\nabla \Phi||_{L^{q*}} \\
&\leq C\varepsilon_1(||\nabla u||_{L^{2p}} + ||\nabla P||_{L^{2p}}). \qquad (45)
\end{aligned}
$$

Observe that $\frac{1}{2} + \frac{1}{2p} + \frac{1}{q^*} = 1 + \frac{1}{2q} - \frac{1}{m} < 1$.

Next we again use (36) to estimate

$$
\begin{aligned}
|G_P \cdot \nabla u| &\leq C\left((|\nabla^2 u| + |\nabla^2 P|)(|\nabla u| + |\nabla P|) + C(|\nabla u|^3 + |\nabla P|^3)\right)|\nabla u| \\
&\leq C(|\nabla^2 u| + |\nabla^2 P| + |\nabla u|^2 + |\nabla P|^2)(|\nabla u|^2 + |\nabla P|^2).
\end{aligned}
$$

Hence by (27)–(29) again we can bound

$$
\begin{aligned}
III &= \int_B G_P \cdot \nabla u \Phi\,dx \\
&\leq C\left(||\nabla^2 u||_{L^2} + ||\nabla^2 P||_{L^2} + ||\nabla u||_{L^4}^2 + ||\nabla P||_{L^4}^2\right)\left(||\nabla u||_{L^4}^2 + ||\nabla P||_{L^4}^2\right) \\
&\leq C\varepsilon_1(||\nabla u||_{L^4}^2 + ||\nabla P||_{L^4}^2). \qquad (46)
\end{aligned}
$$

Upon integrating by parts, finally, we have

$$
IV = \int_B *d\Delta\xi \cdot P\nabla u \Phi\,dx = \int_B d\Delta\xi \wedge \Phi P du = -\int_B \Delta\xi\, d(\Phi P) \wedge du,
$$

and we can bound this term

$$
\begin{aligned}
IV &\leq C||\nabla^2 \xi||_{L^2}\left(||\nabla u||_{L^{2p}}||\nabla\Phi||_{L^{q*}} + ||\nabla u||_{L^4}||\nabla P||_{L^4}\right) \\
&\leq C\varepsilon_1\left(||\nabla u||_{L^{2p}} + ||\nabla u||_{L^4}^2 + ||\nabla P||_{L^4}^2\right). \qquad (47)
\end{aligned}
$$

Our claim (41) follows upon inserting the bounds (45), (46), and (47) into (44).

Upon scaling the bound (41) we obtain

$$
\begin{aligned}
R^{2p-m} \int_{B_R(x_1)} |f|^p\,dx &\leq C\varepsilon_1\Big(||\nabla u||^p_{L^{2p,m-2p}(B_R(x_1))} + ||\nabla P||^p_{L^{2p,m-2p}(B_R(x_1))} \\
&\qquad + ||\nabla u||^{2p}_{L^{4,m-4}(B_R(x_1))} + ||\nabla P||^{2p}_{L^{4,m-4}(B_R(x_1))}\Big). \qquad (48)
\end{aligned}
$$

We use Propositions 3.2, 4.1, and (21) to bound

$$
\begin{aligned}
&||\nabla u||^{2p}_{L^{2p,m-2p}(B_R(x_1))} \\
&\leq C||\nabla u||^p_{L^{1,m-1}(B_R(x_1))}\left(||\nabla^2 u||^p_{L^{p,m-2p}(B_R(x_1))} + ||\nabla u||^p_{L^{p,m-p}(B_R(x_1))}\right) \\
&\leq C||\nabla u||^p_{L^{p,m-p}(B_{R_0}(x_0))}\left(||\nabla^2 u||^p_{L^{p,m-2p}(B_{R_0}(x_0))} + ||\nabla u||^p_{L^{p,m-p}(B_{R_0}(x_0))}\right) \\
&\leq C||\nabla u||^p_{L^{p,m-p}(B_{R_0}(x_0))}\Psi(u;\gamma^{-1}R_0) \leq C\Psi^2(u;\gamma^{-1}R_0) \leq C\Psi^2(\gamma^{-1}R_0). \qquad (49)
\end{aligned}
$$

Using also (27), similarly we can bound

$$
||\nabla P||^p_{L^{2p,m-2p}(B_R(x_1))} \leq C\Psi(\gamma^{-1}R). \qquad (50)
$$

Again invoking Propositions 3.2 and 4.1 together with (21) and (29), moreover, we find

$$\begin{aligned}
||\nabla u||^{2p}_{L^{4,m-4}(B_R(x_1))} &\le C||\nabla u||^p_{L^{1,m-1}(B_R(x_1))}(||\nabla^2 u||^p_{L^{2,m-4}(B_R(x_1))} + ||\nabla u||^p_{L^{2,m-2}(B_R(x_1))}) \\
&\le C||\nabla u||^p_{L^{p,m-p}(B_R(x_1))}(||\nabla^2 u||^p_{L^{2,m-4}(B_R(x_1))} + ||\nabla u||^p_{L^{4,m-4}(B_R(x_1))}) \\
&\le C||\nabla u||^p_{L^{p,m-p}(B_{R_0}(x_0))}(||\nabla^2 u||^p_{L^{2,m-4}(B_2(0))} + ||\nabla u||^p_{L^{4,m-4}(B_2(0))}) \\
&\le C\varepsilon_1 \Psi(u; R_0) \le C\Psi(\gamma^{-1}R_0), \qquad (51)
\end{aligned}$$

and similarly for P. Then from (48) we obtain

$$R^{2p-m} \int_{B_R(x_1)} |f|^p \, dx \le C\varepsilon_1 \Psi(\gamma^{-1}R_0), \qquad (52)$$

as claimed. □

Lemma 4.3 *For any constant* $0 < \gamma < 1$ *and any* $0 < R_0 < \gamma$ *there holds*

$$\Psi(u; \gamma R_0) \le C_1 \gamma^p \Psi(u; R_0) + C\gamma^{2p-2m} \varepsilon_1 \Psi(\gamma^{-1}R_0) \qquad (53)$$

with a uniform constant C_1 *independent of* $\gamma < 1$.

Proof On $B_R(x_1) \subset B_{R_0}(x_0)$ we split $P\Delta u = f + h$ as in (39) above, where $\Delta h = 0$ in $B_R(x_1)$ and with $f\,\big|_{\partial B_R(x_1)} = 0$.

For $r < R$ then from the Campanato estimates for harmonic functions, as in Giaquinta [4, proof of Theorem III.2.2, p.84 f.], we conclude that

$$\begin{aligned}
\int_{B_r(x_1)} |\Delta u|^p \, dx &\le C \int_{B_r(x_1)} |h|^p \, dx + C \int_{B_r(x_1)} |f|^p \, dx \\
&\le C \left(\frac{r}{R}\right)^m \int_{B_R(x_1)} |h|^p \, dx + C \int_{B_r(x_1)} |f|^p \, dx \\
&\le C \left(\frac{r}{R}\right)^m \int_{B_R(x_1)} |\Delta u|^p \, dx + C \int_{B_R(x_1)} |f|^p \, dx. \qquad (54)
\end{aligned}$$

Fixing $r = \gamma R$ and scaling, from Lemma 4.2 we obtain

$$\begin{aligned}
r^{2p-m} \int_{B_r(x_1)} |\Delta u|^p \, dx &\le C\gamma^{2p} R^{2p-m} \int_{B_R(x_1)} |\Delta u|^p \, dx + C\gamma^{2p-m} R^{2p-m} \int_{B_R(x_1)} |f|^p \, dx \\
&\le C\gamma^{2p} \Psi_2(u; R_0) + C\gamma^{2p-m} \varepsilon_1 \Psi(\gamma^{-1}R_0).
\end{aligned}$$

Also passing to the supremum with respect to $B_R(x_1) \subset B_{R_0}(x_0)$ on the left hand side, we thus find

$$\Psi_2(u; \gamma R_0) \le C\gamma^{2p} \Psi_2(u; R_0) + C\gamma^{2p-m} \varepsilon_1 \Psi(\gamma^{-1}R_0). \qquad (55)$$

Similarly, we split $u = u_0 + u_1$ on $B_R(x_1) \subset B_{R_0}(x_0)$, where $\Delta u_0 = 0$ and with $u_1 = 0$ on $\partial B_R(x_1)$. As above then we obtain

$$\int_{B_r(x_1)} |\nabla u|^p \, dx \le C \left(\frac{r}{R}\right)^m \int_{B_R(x_1)} |\nabla u|^p \, dx + C \int_{B_R(x_1)} |\nabla u_1|^p \, dx. \qquad (56)$$

But since $u_1 \in W_0^{1,p}(B_R(x_1))$ with $\Delta u_1 = \Delta u \in L^p(B_R(x_1))$, the Calderòn–Zygmund inequality yields that

$$\int_{B_R(x_1)} |\nabla u_1|^p \, dx \le CR^p \int_{B_R(x_1)} |\Delta u|^p \, dx. \tag{57}$$

Upon scaling, for $r = \gamma R$ we thus find the inequality

$$\begin{aligned} r^{p-m} \int_{B_r(x_1)} |\nabla u|^p \, dx &\le C\gamma^p R^{p-m} \int_{B_R(x_1)} |\nabla u|^p \, dx + C\gamma^{p-m} C R^{2p-m} \int_{B_R(x_1)} |\Delta u|^p \, dx \\ &\le C\gamma^p \Psi(u; R_0). \end{aligned}$$

After passing to the supremum with respect to $B_R(x_1) \subset B_{R_0}(x_0)$, similar to (55) then we obtain

$$\Psi_1(u; \gamma R_0) \le C\gamma^p \Psi(u; R_0). \tag{58}$$

Since $\gamma < 1$ we may combine (54) and (58) to deduce the bound

$$\begin{aligned} \Psi(u; \gamma R_0) &= \Psi_1(u; \gamma R_0) + \gamma^{-m} \Psi_2(u; \gamma R_0) \\ &\le C_1 \gamma^p \Psi(u; R_0) + C\gamma^{2p-2m} \varepsilon_1 \Psi(\gamma^{-1} R_0) \end{aligned} \tag{59}$$

with a uniform constant C_1 independent of $\gamma < 1$. □

Lemma 4.4 *For any constant* $0 < \gamma < 1$ *and any* $0 < R_0 < \gamma$ *there holds*

$$\Psi(P; \gamma R_0) \le C_2 \gamma^p \Psi(P; R_0) + C_2 \Psi(u; \gamma R_0) + C\varepsilon_1 \gamma^{-m} \Psi(\gamma^{-1} R_0) \tag{60}$$

with a uniform constant $C_2 \ge 1$ *independent of* $\gamma < 1$.

Proof Recalling the definition (16) of Ω, we see that

$$|d * \Omega| \le C(|du|^2 + |\Delta u|).$$

From (17) and (25) then it follows that

$$\begin{aligned} |\Delta P| = |d * dP| &= |d * (dP P^{-1}) + (-1)^m * dP \wedge dP^{-1}| \\ &\le |dP|^2 + |d * (P\Omega P^{-1})| \le C(|dP|^2 + |dP||\Omega| + |d * \Omega|) \\ &\le C(|dP|^2 + |du|^2 + |\Delta u|) \end{aligned}$$

and

$$\|\Delta P\|_{L^p} \le C(\|dP\|^2_{L^{2p}} + \|du\|^2_{L^{2p}} + \|\Delta u\|_{L^p}).$$

Using (49) and (50), with a constant C_2 independent of $0 < \gamma < 1$ we may bound

$$\begin{aligned} \Psi_2(P; \gamma R_0) &\le C\|\Delta u\|^p_{L^{p,m-2p}(B_{\gamma R_0}(x_0))} + C\|dP\|^{2p}_{L^{2p,m-2p}(B_{\gamma R_0}(x_0))} \\ &\quad + C\|du\|^{2p}_{L^{2p,m-2p}(B_{\gamma R_0}(x_0))} \\ &\le C_2 \Psi_2(u; \gamma R_0) + C\Psi^2(R_0) \le C_2 \Psi_2(u; \gamma R_0) + C\varepsilon_1 \Psi(\gamma^{-1} R_0). \end{aligned}$$

Possibly choosing a larger constant C_2, moreover, similar to (58) we have

$$\Psi_1(P; \gamma R_0) \le C\gamma^p \Psi_1(P; R_0) + C\gamma^{p-m} \Psi_2(P; R_0) \le C_2 \gamma^p \Psi(P; R_0). \tag{61}$$

□

Combining (53), (60), and Lemma 4.2, we deduce the bound

$$\Psi(P;\gamma R_0)+2C_2\Psi(u;\gamma R_0)\le C_3\gamma^p\Psi(R_0)+C_2\Psi(u;\gamma R_0)+C\gamma^{2p-2m}\varepsilon_1\Psi(\gamma^{-1}R_0), \tag{62}$$

where the constant C_3 is independent of $0<\gamma<1$. With our choice of

$$\Psi(r)=\Psi(P;r)+C_2\Psi(u;r)$$

it follows that for all $R_0<\gamma$ we have

$$\begin{aligned}\Psi(\gamma R_0)&\le C_3\gamma^p\Psi(R_0)+C\gamma^{2p-2m}\varepsilon_1\Psi(\gamma^{-1}R_0)\\&\le C_3\gamma^p(1+C_4\gamma^{p-2m}\varepsilon_1)\Psi(\gamma^{-1}R_0)\end{aligned} \tag{63}$$

with a constant C_4 possibly depending on γ. That is, for all $R_1<1$ there holds

$$\Psi(\gamma^2R_1)\le C_3\gamma^p(1+C_4\gamma^{p-2m}\varepsilon_1)\Psi(R_1). \tag{64}$$

Choose $0<\gamma<1$ such that $2C_3\gamma^{p/2}=1$ and let $\varepsilon_1>0$ be such that $C_4\gamma^{p-2m}\varepsilon_1=1$. Letting $\delta=\gamma^2<1$, then for any $R<1$ we find

$$\Psi(\delta R)\le 2C_3\gamma^p\Psi(R)=\gamma^{p/2}\Psi(R)=\delta^{p/4}\Psi(R). \tag{65}$$

For any $0<r\le\delta$ determine $k\in\mathbb{N}$ such that $\delta^{k+1}<r\le\delta^k$. From (65) then by iteration we obtain

$$\Psi(r)\le\Psi(\delta^k)\le\delta^{p/4}\Psi(\delta^{k-1})\le\cdots\le\delta^{kp/4}\Psi(1)\le Cr^{p/4}\Psi(1)\le Cr^{p/4}, \tag{66}$$

as desired. The proof is complete.

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Appendix A

For completeness, here we present the proof of Proposition 3.2. Clearly we may suppose that $B=B_1(0)$. Given $u\in L_2^{s,m-2s}(B)$ we may extend u to a function $v\in L_2^{s,m-2s}(B_2(0))$ with

$$\begin{aligned}||\nabla v||^2_{L^{p,m-p}(B_2(0))}&\le C||\nabla u||^2_{L^{p,m-p}(B)},\\ ||\nabla^2 v||_{L^{s,m-2s}(B_2(0))}&\le C||\nabla^2 u||_{L^{s,m-2s}(B)}\end{aligned} \tag{67}$$

for all $1\le p\le s$ with some constant $C=C(m)$ independent of u. Shifting v by a constant, if necessary, we may assume that $\bar v_{1,0}=0$. Let $\varphi\in C_0^\infty(B_2(0))$ be a smooth cut-off function with $0\le\varphi\le 1$ and such that $\varphi\equiv 1$ on $B_1(0)$. Applying Proposition 3.1 to the function $w=v\varphi\in L_2^{s,m-2s}(\mathbb{R}^m)$ and using (23), we obtain

$$\begin{aligned}||\nabla u||^2_{L^{2s,m-2s}(B)}&\le C[w]_{BMO(\mathbb{R}^m)}||\nabla^2 w||_{L^{s,m-2s}(\mathbb{R}^m)}\\&\le C||\nabla w||_{L^{1,m-1}(\mathbb{R}^m)}||\nabla^2 w||_{L^{s,m-2s}(\mathbb{R}^m)}.\end{aligned} \tag{68}$$

The claimed estimate thus follows if we can bound

$$||\nabla w||_{L^{1,m-1}(\mathbb{R}^m)}\le C||\nabla u||_{L^{1,m-1}(B)} \tag{69}$$

and

$$||\nabla^2 w||_{L^{s,m-2s}(\mathbb{R}^m)}\le C\left(||\nabla^2 u||_{L^{s,m-2s}(B)}+||\nabla u||_{L^{s,m-s}(B)}\right). \tag{70}$$

For $B_R(x_1) \subset B_2(0)$ we estimate

$$|\nabla w| \le C\left(|\nabla v| + |v - \bar{v}_{R,x_1}| + |\bar{v}_{R,x_1}|\right)$$

and use (22) to obtain

$$\int_{B_R(x_1)} |\nabla w|\,dx \le C \int_{B_R(x_1)} |\nabla v|\,dx + CR^m|\bar{v}_{R,x_1}|.$$

From (67) we conclude that

$$||\nabla w||_{L^{1,m-1}(\mathbb{R}^m)} \le C||\nabla u||_{L^{1,m-1}(B)} + C \sup_{B_R(x_1)\subset B_2(0)} R|\bar{v}_{R,x_1}|. \tag{71}$$

But for $B_{R/2}(x_2) \subset B_R(x_1)$ we can estimate

$$\begin{aligned}|\bar{v}_{R,x_1} - \bar{v}_{R/2,x_2}| &= \left|\ \fint_{B_{R/2}(x_2)} (\bar{v}_{R,x_1} - v)\,dx\right| \\ &\le C \fint_{B_R(x_1)} |\bar{v}_{R,x_1} - v|\,dx \le CR^{1-m} \int_{B_R(x_1)} |\nabla v|\,dx \\ &\le C||\nabla v||_{L^{1,m-1}(B_2(0))} \le C||\nabla u||_{L^{1,m-1}(B)}.\end{aligned}$$

Hence for any $B_R(x_1) \subset B_2(0)$ we can bound

$$|\bar{v}_{R,x_1}| \le C|\log R|||\nabla u||_{L^{1,m-1}(B)} + |\bar{v}_{1,0}| = C|\log R|||\nabla u||_{L^{1,m-1}(B)} \tag{72}$$

and (69) follows from (71).

For $B_R(x_1) \subset B_2(0)$ similarly we estimate

$$|\nabla^2 w| \le C\left(|\nabla^2 v| + |\nabla v| + |v - \bar{v}_{R,x_1}| + |\bar{v}_{R,x_1}|\right)$$

to conclude the bound

$$||\nabla^2 w||_{L^{s,m-2s}(\mathbb{R}^m)} \le C\left(||\nabla^2 v||_{L^{s,m-2s}(B_2(0))} + ||\nabla v||_{L^{s,m-s}(B_2(0))} + \sup_{B_R(x_1)\subset B_2(0)} R^2|\bar{v}_{R,x_1}|\right),$$

and (70) follows from (67) and (72).

Appendix B

Assume that condition (3) is satisfied on $B_3(0)$. To show the Morrey bound (29) it suffices to show that at every Lebesgue point $x_0 \subset B_2(0)$ of the function $|\nabla^2 u|^2 + |\nabla u|^2$ for any $0 < r < 1$ and some radius $r/2 < \rho < r$ there holds

$$\rho^{4-m} \int_{B_\rho(x_0)} |\Delta u|^2\,dx + \rho^{3-m} \int_{\partial B_\rho(x_0)} |\nabla u|^2\,do \le C\varepsilon_0 \tag{73}$$

with a constant $C = C(N, m)$. Indeed, by elliptic regularity theory the bound (73) implies that $u \in H^{3/2}(B_\rho(x_0)) \cap H^2_{\mathrm{loc}}(B_\rho(x_0))$ with

$$r^{4-m} \int\limits_{B_{r/3}(x_0)} |\nabla^2 u|^2 \, dx + r^{2-m} \int\limits_{B_{r/3}(x_0)} |\nabla u|^2 \, dx$$
$$\leq C\rho^{4-m} \int\limits_{B_\rho(x_0)} |\Delta u|^2 \, dx + \rho^{3-m} \int\limits_{\partial B_\rho(x_0)} |\nabla u|^2 \, do \leq C\varepsilon_0. \tag{74}$$

Since N is compact, we also have $|u| \leq C(N)$ almost everywhere and (29) follows from interpolating

$$\int\limits_{B_{r/4}(x_0)} |\nabla u|^4 \, dx \leq C \sup_{B_{r/3}(x_0)} |u|^2 \int\limits_{B_{r/3}(x_0)} (|\nabla^2 u|^2 + r^{-2}|\nabla u|^2) dx. \tag{75}$$

To see (73) fix a Lebesgue point $x_0 \subset B_2(0)$ as above. After a shift of coordinates we may assume that $x_0 = 0$. Also let $B_r = B_r(0)$ for brevity. Using the notation $u_\alpha = \partial_\alpha u$, etc., we may write the monotonicity formula of [3], Proposition 3.2, in the form

$$\sigma(r) - \sigma(\rho) = \int\limits_{B_r \setminus B_\rho} \left(\frac{|u_\beta + x^\alpha u_{\alpha\beta}|^2}{|x|^{m-2}} + (m-2)\frac{|x^\alpha u_\alpha|^2}{|x|^m} \right) dx, \tag{76}$$

where $\sigma(r) = \sigma_1(r) + \sigma_2(r)$ with

$$\sigma_1(r) = r^{4-m} \int\limits_{B_r} |\Delta u|^2 \, dx + r^{3-m} \int\limits_{\partial B_r} |\nabla u|^2 \, do \tag{77}$$

and

$$\sigma_2(r) = r^{3-m} \int\limits_{\partial B_r} (2x^\alpha u_{\alpha\beta} u_\beta + 3|\nabla u|^2 - 4r^{-2}|x^\alpha u_\alpha|^2) \, do. \tag{78}$$

Note that for a *"good"* radius $r > 0$ we can bound

$$|\sigma(r)| \leq Cr^{4-m} \int\limits_{B_r} |\Delta u|^2 \, dx + Cr^{5-m} \int\limits_{\partial B_r} (|\nabla^2 u|^2 + r^{-2}|\nabla u|^2) \, do$$
$$\leq Cr^{4-m} \int\limits_{B_{2r}} (|\nabla^2 u|^2 + r^{-2}|\nabla u|^2) dx. \tag{79}$$

Since we assume that $x_0 = 0$ is a Lebesgue point for the function $|\nabla^2 u|^2 + |\nabla u|^2$ we then conclude that

$$\liminf_{r \downarrow 0} |\sigma(r)| = 0. \tag{80}$$

Moreover, from (3) we have $|\sigma(1)| \leq C\varepsilon_0$. Hence from (76) we deduce the bound

$$\int\limits_{B_1} \left(\frac{|u_\beta + x^\alpha u_{\alpha\beta}|^2}{|x|^{m-2}} + (m-2)\frac{|x^\alpha u_\alpha|^2}{|x|^m} \right) dx \leq C\varepsilon_0. \tag{81}$$

For any $r < 1$ then we have

$$\inf_{r/2<\rho<r} \rho^{3-m} \int_{\partial B_\rho} (|u_\beta + x^\alpha u_{\alpha\beta}|^2 + 4\rho^{-2}|x^\alpha u_\alpha|^2)\, do$$

$$\leq C \int_{B_r \setminus B_{r/2}} \left(\frac{|u_\beta + x^\alpha u_{\alpha\beta}|^2}{|x|^{m-2}} + (m-2)\frac{|x^\alpha u_\alpha|^2}{|x|^m} \right) dx \leq C\varepsilon_0. \tag{82}$$

But estimating

$$2x^\alpha u_{\alpha\beta} u_\beta + 3|\nabla u|^2 = 2(u_\beta + x^\alpha u_{\alpha\beta})u_\beta + |\nabla u|^2 \geq -|u_\beta + x^\alpha u_{\alpha\beta}|^2, \tag{83}$$

we can bound

$$\sigma_2(\rho) \geq -\rho^{3-m} \int_{\partial B_\rho} (|u_\beta + x^\alpha u_{\alpha\beta}|^2 + 4\rho^{-2}|x^\alpha u_\alpha|^2)\, do, \tag{84}$$

and from (82) we conclude that

$$\sup_{r/2<\rho<r} \sigma_2(\rho) \geq -C\varepsilon_0. \tag{85}$$

For a suitable radius $r/2 < \rho < r$ the monotonicity estimate (76) then yields the bound

$$\sigma_1(\rho) \leq \sigma(1) - \sigma_2(\rho) \leq C\varepsilon_0; \tag{86}$$

that is, we have (73), as desired.

Observe that in contrast to [3, Lemma 4.8], or [14, Lemma 5.3], we do not use the biharmonic map equation.

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